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GAME-THEORETIC SOLUTIONS FOR SOME ECONOMIC SITUATIONS

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1. INTRODUCTION

Many economic situations, where cooperation between persons, firms, institutions, etc. is possible, can be modeled as a cooperative game in characteristic function form. For these games, various solution concepts have been introduced which prescribe the distribution of the total gain.

In this paper we consider a production economy, in which several landless peasants and one (or two) landowner(s) are involved. This particular production model (mainly with one landowner) has been studied already by Shapley, Shubik [8] and Chetty, Dasgupta, Raghavan [1]. In [8] the core and the Shapley value of the corresponding cooperative game are emphasized, while in [1] the nucleolus of the game is central. We will emphasize the τ-value of the game and compare it with the other three solution concepts.

For this economic model, we assume that the two landowners are identical. The peasants have nothing to contribute but their labour and are also assumed to be identical. The landowners hire peasants to cultivate their land. If t peasants are hired by one landowner, then the monetary value of the crop of the land, cultivated by those t peasants, is denoted by $f(t)$. The function $f : \{0, 1, \ldots, n\} \to \mathbb{R}$ is called the *production function,* where n is the number of peasants. Throughout this paper it is assumed that

$$f(t) \geq 0 \text{ for all } t \in \{1, 2, \ldots, n\} \tag{1.1}$$

$$f(0) = 0, \text{ i.e. a landowner by himself can not produce anything} \tag{1.2}$$

$$f(t+1) \geq f(t) \text{ for all } t \in \{0, 1, \ldots, n-1\}. \tag{1.3}$$

We look at the two cases in which the function f is convex or concave. The production function $f : \{0, 1, \ldots, n\} \to \mathbb{R}$ is said to be

convex if $f(t+1) - f(t) > f(t) - f(t-1)$ for all $t \in \{1, 2, \ldots, n-1\}$,
concave if $f(t+1) - f(t) < f(t) - f(t-1)$ for all $t \in \{1, 2, \ldots, n-1\}$.

Thus, f is convex (concave) iff the marginal returns of the function f form an increasing (decreasing) sequence. In the next sections we use the following properties of such functions.

If f is convex, then (1.4)

(i) $(f(p+1) - f(p))\,(r-t) \leq f(r) - f(t) \leq (r-t)\,(f(s) - f(s-1))$ for all $0 \leq p \leq t < r \leq s \leq n$

(ii) $t^{-1}f(t) \leq (t+1)^{-1}\, f(t+1)$ for all $1 \leq t \leq n-1$

If f is concave, then (1.5)

$(f(s) - f(s-1))\ (r-t) \leqslant f(r) - f(t) \leqslant (r-t)\ (f(p+1) - f(p))$ for all $0 \leqslant p \leqslant t < r \leqslant s \leqslant n$.

The game-theoretic solutions for this economic model are treated in the sections 3 and 4. In section 2 we recall the definitions of several game-theoretic notions and solution concepts.

2. SOLUTION CONCEPTS FOR COOPERATIVE GAMES

A *cooperative n-person game in characteristic function form* is a pair (N,v), where $N := \{1, 2, \ldots, n\}$ with $n \geqslant 2$ and v is a real-valued function on the family 2^N of all subsets of N satisfying $v(\emptyset) = 0$. N is the set of *players* and v the *characteristic function* of the game. A subset S of N is called a *coalition* and $v(S)$ the *worth* of the coalition S in the game. The function v itself will also be called an n-person game. We denote the set of all n-person games by G^n.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $S \subset N$. The i-th coordinate x_i of the vector x represents the payoff to player $i \in N$. We often write $x(S)$ instead of $\sum_{i \in S} x_i$, where $x(\emptyset) := 0$. The payoff vector x is said to be *efficient* for $v \in G^n$ if $x(N) = v(N)$.

The *core* $C(v)$ of a game $v \in G^n$ is defined as the set of all efficient payoff vectors that can not be improved upon by any coalition. That is for any $v \in G^n$,

$$C(v) := \{x \in \mathbb{R}^n;\ x(N) = v(N) \ \text{ and } \ x(S) \geqslant v(S) \text{ for all } S \subset N\}. \tag{2.1}$$

We denote the set of n-person games with a non-empty core by B^n. Concerning the *nucleolus* $\eta(v) \in \mathbb{R}^n$ of a game $v \in G^n$, introduced by Schmeidler [6], we only recall that it possesses a "central" location within the core whenever the core of the game v is not empty (Maschler, Peleg, Shapley [5]).

The *Shapley value* $\phi(v) \in \mathbb{R}^n$ of a game $v \in G^n$, introduced by Shapley [7], can be defined by the formula

$$\phi_i(v) := n^{-1} \sum_{s=1}^{n} (c(n,s))^{-1} \sum_{S;\, i \in S,\, |S|=s} [v(S) - v(S - \{i\})] \tag{2.2}$$

where $i \in N$ and $c(n,s)$ is the number of coalitions of size s containing the designated player i :

$$c(n,s) := \binom{n-1}{s-1} = (n-1)!/(s-1)!(n-s)!. \tag{2.3}$$

By (2.2), the Shapley value $\phi_i(v)$ of player i in the game v is his average marginal worth to all the coalitions to which he might join where a coalition of size $s-1$ has weight $(n\ c(n,s))^{-1}$.

Finally we recall the definition of the τ-value, a solution concept for n-person games introduced by Tijs [9]. In the definition of the τ-value play a role the vectors b^v, λ^v and the map $g^v : 2^N \to \mathbb{R}$. For any $v \in G^n$, these notions are defined by

$$b_i^v := v(N) - v(N - \{i\}) \qquad \text{for all } i \in N, \tag{2.4}$$

$$g^v(S) := \sum_{i \in S} b_i^v - v(S) = b^v(S) - v(S) \qquad \text{for all } S \subset N, \tag{2.5}$$

$$\lambda_i^v := \min_{S;\, i \in S} g^v(S) \qquad \text{for all } i \in N. \tag{2.6}$$

The i-th coordinate b_i^v of b^v can be seen as the *marginal contribution of player* i to the grand coalition N in the game v.
Tijs, Driessen [10] proved the following result : for any $v \in G^n$,

$$b_i^v - \lambda_i^v \leq x_i \leq b_i^v \text{ for each } x \in C(v) \text{ and all } i \in N. \tag{2.7}$$

This means that the vector $b^v - \lambda^v$ (b^v) is a lower (upper) bound for the core of the game v. Hence, it is called the *lower (upper) vector* of the game v. The vector λ^v itself is called the *concession vector* of v. The map g^v is said to be the *gap function* of the game v and $g^v(S)$ the *gap of the coalition* S in v.
For a game $v \in B^n$ with non-empty core, the *τ-value* $\tau(v) \in \mathbb{R}^n$ is defined as the unique efficient payoff vector for the game v, lying on the line segment $[b^v - \lambda^v, b^v]$ with end points the lower vector $b^v - \lambda^v$ and the upper vector b^v of the game v. The formula for the τ-value $\tau(v)$ of a game $v \in B^n$ can be given by

$$\begin{aligned} \tau(v) &= b^v && \text{if } g^v(N) = 0 \\ &= b^v - g^v(N)\,(\lambda^v(N))^{-1}\,\lambda^v && \text{if } g^v(N) > 0 \end{aligned} \tag{2.8}$$

Although the τ-value of a game $v \in B^n$ is defined with the aid of bounds for the core, it is not necessarily a core-element. For the definition of the τ-value of an arbitrary game $v \in G^n$, we refer to Driessen, Tijs [4].

The three solution concepts nucleolus, Shapley value and τ-value possess many nice properties, e.g. efficiency and symmetry. These two properties will be used in order to determine these three solution concepts for the game, arising from the economic model mentioned in section 1.

3. *THE PRODUCTION ECONOMY WITH ONE LANDOWNER AND n PEASANTS*

In this section we consider the situation in which only one landowner and n landless peasants are involved ($n \geq 1$). We regard the landowner as player 1 and the peasants as players 2, ..., n+1. Then this situation corresponds to the (n+1)-person game (N,v) where $N = \{1, 2, \ldots, n+1\}$ and the characteristic function $v : 2^N \to \mathbb{R}$ is defined by

$$\begin{aligned} v(S) &= 0 && \text{if } 1 \notin S \\ &= f(s-1) && \text{if } 1 \in S \text{ (where } s = |S|). \end{aligned} \tag{3.1}$$

The worth of any coalition of peasants alone equals zero since they do not own any land. Because $f(0) = 0$, we have also that $v(\{i\}) = 0$ for $i \in N$.

In the following we summarize and discuss the outcomes of the four solution concepts, which are mentioned in section 2, for the game v defined by (3.1). First of all we consider the core of the game v. Particularly, we are interested in the symmetrical part of the core. A core-element $x = (x_1, x_2, \ldots, x_{n+1}) \in C(v)$ of the game v is said to be *symmetrical* if $x_i = x_j$ for all $i,j \in N - \{1\}$. The set of the symmetrical core-elements of the game v is denoted by SC(v). Note that SC(v) is a convex set and that in general $SC(v) \subsetneq C(v)$. Furthermore, by (2.4), $b^v = (f(n), \Delta, \ldots, \Delta) \in \mathbb{R}^{n+1}$ where $\Delta := f(n) - f(n-1)$. Note that $\Delta \geq 0$ by assumption (1.3).

Theorem 3.1. Let v be the game defined by (3.1). Then we have :

(i) $C(v) = \{x \in \mathbb{R}^{n+1};\ x(N) = f(n),\ x_i \geq 0$ for all $i \in N$ and

$x(S) \geq f(s-1)$ for all $S \subset N$ with $1 \in S,\ s \geq 2\}$

(ii) $SC(v) = \{(f(n) - n\beta, \beta, \ldots, \beta) \in \mathbb{R}^{n+1};\ 0 \leq \beta \leq n^{-1}f(n)$ and

$f(n) - f(t) \geq (n-t)\beta$ for all $t \in \{1, \ldots, n\}\}$.

(iii) If f is convex, then

$SC(v) = \text{co}\ \{(f(n), 0, \ldots, 0), (0, n^{-1}f(n), \ldots, n^{-1}f(n))\}$.

(iv) If f is concave, then

$SC(v) = \text{co}\ \{(f(n), 0, \ldots, 0), (f(n) - n\Delta, \Delta, \ldots, \Delta)\}$.

Proof.

(i) follows immediately from the defintions (2.1) and (3.1).

(ii) is a direct consequence of (i) and the definition of SC(v).

(iii) By (ii) and the assumptions (1.1), (1.3), we have that $(f(n), 0, \ldots, 0) \in SC(v)$ for any production function f. By (ii), (1.1) and (1.4) (ii), we have also that $(0, n^{-1}f(n), \ldots, n^{-1}f(n)) \in SC(v)$. It follows now from (ii) and the convexity of SC(v), that (iii) holds.

(iv) For any $(f(n) - n\beta, \beta, \ldots, \beta) \in SC(v)$, we have by (ii) that $0 \leq \beta \leq f(n)-f(n-1) = \Delta$ Using (ii) and (1.5), we have also that $(f(n) - n\Delta, \Delta, \ldots, \Delta) \in SC(v)$. It follows now from (ii) and the convexity of SC(v), that (iv) holds. □

In the cases that the production function f is convex or concave, we are able to get nice expressions for the nucleolus and the τ-value of the game v. For any production function f, the Shapley value of the game v appears to be an average output according to the next theorem.

Theorem 3.2. Let v be the game defined by (3.1). Then we have

(i) (Shapley, Shubik [8]) $\phi_1(v) = (n+1)^{-1} \sum_{t=0}^{n} f(t)$ and

$\phi_j(v) = n^{-1}(f(n) - \phi_1(v))$ for all $j \in N - \{1\}$.

(ii) (Chetty, Dasgupta, Raghavan [1])

If f is concave, then $\eta(v) = (f(n) - \frac{1}{2}n\Delta, \frac{1}{2}\Delta, \ldots, \frac{1}{2}\Delta) \in \mathbb{R}^{n+1}$.

If f is convex, then

$\eta(v) = (f(n) - \frac{1}{2}n\Delta, \frac{1}{2}\Delta, \ldots, \frac{1}{2}\Delta)$ whenever $(n+1)^{-1} f(n) \geq \frac{1}{2}\Delta$

$= (n+1)^{-1} f(n)\ (1, 1, \ldots, 1)$ whenever $(n+1)^{-1} f(n) \leq \frac{1}{2}\Delta$.

(iii) (Driessen, Tijs [3]) If f is concave, then $\tau(v) = \eta(v) \in C(v)$.

(Driessen, Tijs [2]) If f is convex, then

$\tau(v) = (n\Delta + f(n))^{-1}f(n)\ \ (f(n), \Delta, \ldots, \Delta) \in C(v)$.

Let f be concave. In view of the theorems 3.1(iv) and 3.2(ii), (iii), both the nucleolus and the τ-value coincide with the center of the symmetrical core. Particularly, the τ-value is in the core while the Shapley value not necessarily belongs to the core. If the τ-value (or nucleolus) is the payoff vector for the game, then any peasant receives half of his marginal contribution Δ while the landowner gets at least the sum of the payoffs to the peasants (since $f(n) \geq n\Delta$ by (1.5)).

Let f be convex. In view of theorem 3.2(ii) and (iii), the nucleolus differs from the τ-value. Nevertheless, $\tau(v) \in C(v)$ because $\tau(v) = (n\Delta + f(n))^{-1}f(n)\ (f(n), \Delta, \ldots, \Delta) = (n\Delta + f(n))^{-1}f(n)\ (f(n), 0, \ldots, 0) + (n\Delta + f(n))^{-1}\ n\Delta(0, n^{-1}f(n), \ldots, n^{-1}f(n)) \in SC(v)$ by the theorems 3.2(iii) and 3.1(iii). Similarly, it can be shown that $\phi(v) \in C(v)$. In general the Shapley value, the τ-value and the nucleolus will not coincide with the center of the symmetrical core. If the τ-value is the payoff vector for the game, then any peasant receives a part of his marginal contribution. Note that this part is less than one half since $(n\Delta + f(n))^{-1}f(n) \leqslant \frac{1}{2}$ by (1.4)(i). Furthermore, the landowner gets at most the sum of the payoffs to the peasants, but at least as much as a peasant. If the nucleolus is the payoff vector, then any peasant gets half of his marginal contribution Δ if and only if this payoff to any peasant does not exceed the amount which is left for the landowner. Otherwise, the total gain f(n) is distributed equally among all players.

4. THE PRODUCTION ECONOMY WITH TWO LANDOWNERS AND n PEASANTS

In this section we study the situation with two identical landowners (players 1 and 2) and n identical landless peasants (players 3, ..., n+2). The corresponding (n+2)-person game (N,w) is defined as follows :

$$N = \{1, 2, 3, \ldots, n+2\} \text{ with } n \geqslant 1 \text{ and}$$

$$\begin{aligned} w(S) &= 0 && \text{if } 1 \notin S \text{ and } 2 \notin S \\ &= f(s-1) && \text{if } 1 \in S,\ 2 \notin S \text{ or } 1 \notin S,\ 2 \in S \\ &= \max\{f(r) + f(t);\ r, t \in \mathbb{N},\ r+t = s-2\} && \text{if } \{1, 2\} \subset S. \end{aligned} \tag{4.1}$$

Note that, if the two landowners are both participants of a coalition S, then they cooperate in order to attain the largest possible output from S by dividing the peasants of S into two disjoint groups in the most efficient way. Furthermore, $w(\{i\}) = 0$ for all $i \in N$.

This economic situation, in which *two* landowners are involved, has not been studied by Shapley, Shubik [8] while Chetty, Dasgupta, Raghavan [1] have only calculated bounds for the nucleolus $\eta(w)$ of the game w in case n is even and f is concave. We determine the core C(w), the nucleolus $\eta(w)$, the τ-value $\tau(w)$ and the Shapley value $\phi(w)$ of the game w in the cases that f is convex or concave. As in the previous section, we are also interested in the symmetrical part SC(w) of the core of w. It consists of those core-elements $x = (x_1, x_2, x_3, \ldots, x_{n+2}) \in C(w)$ of the game w, which satisfy $x_1 = x_2$ and $x_i = x_j$ for all $i,j \in N - \{1, 2\}$.

A. Throughout this subsection we assume that f is *convex*. It follows from (1.2) and (1.4)(i) that

$$w(S) = f(s-2) \text{ whenever } \{1, 2\} \subset S \tag{4.2}$$

Hence, by (2.4), $b^w = (0, 0, \Delta, \ldots, \Delta) \in \mathbb{R}^{n+2}$ where $\Delta = f(n) - f(n-1)$. In the next theorem we describe the solution concepts.

<u>*Theorem 4.1*</u>. Let w be the game defined by (4.1) where f is convex. Then we have

(i) $C(w) = \{(0, 0, x_3, x_4, \ldots, x_{n+2}) \in \mathbb{R}^{n+2};\ \sum_{i=3}^{n+2} x_i = f(n)$ and $x(S) \geqslant f(s)$ for all $\emptyset \neq S \subset \{3, 4, \ldots, n+2\}\}$

(ii) $SC(w) = \{(0, 0, n^{-1}f(n), \ldots, n^{-1}f(n))\}$

(iii) $\tau(w) = \eta(w) = (0, 0, n^{-1}f(n), \ldots, n^{-1}f(n)) \in SC(w)$

(iv) $\phi_1(w) = \phi_2(w) = ((n+2)(n+1))^{-1} \sum_{t=0}^{n} (n+1-t)\, f(t)$ and

$\phi_i(w) = n^{-1}(f(n) - 2\phi_1(w))$ for all $i \in N - \{1, 2\}$.

<u>Proof</u>.

(i) Let $x \in C(w)$. Then $x_1 \geq w(\{1\}) = 0$ and by (2.7), $x_1 \leq b_1^w = 0$. Thus $x_1 = 0$. Similarly, $x_2 = 0$. In view of (4.2), the definitions (2.1), (4.1) and the assumption (1.1), it is now straightforward to show that (i) is valid.

(ii) follows immediately from (i) by using property (1.4)(ii).

(iii) Because the nucleolus is always contained in the core and possesses the symmetry property, we have $\eta(w) \in SC(w)$. By (2.5), (2.6) and the definition of the τ-value, we have also that $w(\{i\}) \leq b_i^w - \lambda_i^w \leq \tau_i(w) \leq b_i^w$ for all $i \in N$. But $w(\{j\}) = 0 = b_j^w$ for $j \in \{1, 2\}$, so $\tau_1(w) = \tau_2(w) = 0$. It follows now from the symmetry and efficiency that $\tau_i(w) = n^{-1}f(n)$ for all $i \in N - \{1, 2\}$. In view of (ii), we see that (iii) holds.

(iv) follows by straightforward calculations from (2.2) and (2.3). □

If the τ-value or nucleolus is the payoff vector for the game w, then the landowners get nothing. The Shapley value, however, assigns a positive amount to the landowners, which implies that the Shapley value is not in the core.

B. Throughout this subsection we assume that f is *concave*. By (1.5) we have that

$$\begin{aligned} w(S) &= 2f(\tfrac{s}{2} - 1) && \text{if } \{1, 2\} \subset S \text{ and } s \text{ is even} \\ &= f(\tfrac{s-1}{2}) + f(\tfrac{s-3}{2}) && \text{if } \{1, 2\} \subset S \text{ and } s \text{ is odd} \end{aligned} \tag{4.3}$$

The next theorem states that the core itself is symmetrical and hence, it is a line segment (or a single point). The nucleolus is just the midpoint of this line segment.

<u>*Theorem 4.2*</u>. Let w be the game defined by (4.1) where f is concave. Then we have :

(i) $C(w) = SC(w)$

(ii) If n is odd, then $C(w) = \{(\frac{1}{2} w(N) - \frac{1}{2} n\delta, \frac{1}{2} w(N) - \frac{1}{2} n\delta, \delta, \ldots, \delta)\}$ where $w(N) = f(\frac{n+1}{2}) + f(\frac{n-1}{2})$ and $\delta := f(\frac{n+1}{2}) - f(\frac{n-1}{2})$.

(iii) If n is even, then

$C(w) = \text{co}\,\{(f(\frac{1}{2} n) - \frac{1}{2} n\delta^+, f(\frac{1}{2} n) - \frac{1}{2} n\delta^+, \delta^+, \ldots, \delta^+), (f(\frac{1}{2} n) - \frac{1}{2} n\delta^-, f(\frac{1}{2} n) - \frac{1}{2} n\delta^-, \delta^-, \ldots, \delta^-)\}$

where $\delta^+ := f(\frac{1}{2}n + 1) - f(\frac{1}{2} n)$ and $\delta^- := f(\frac{1}{2} n) - f(\frac{1}{2}n - 1)$.

(iv) If n is odd, then $\eta(w) = (\frac{1}{2} w(N) - \frac{1}{2} n\delta, \frac{1}{2} w(N) - \frac{1}{2} n\delta, \delta, \ldots, \delta)$.

If n is even, then $\eta(w) = (f(\frac{1}{2} n) - \frac{1}{4} n\delta^*, f(\frac{1}{2} n) - \frac{1}{4} n\delta^*, \frac{1}{2} \delta^*, \ldots, \frac{1}{2} \delta^*)$

where $\delta^* := \delta^+ + \delta^- = f(\frac{1}{2}n + 1) - f(\frac{1}{2}n - 1)$.

<u>Proof</u>.

(i) Let $P := \{3, \ldots, n+2\}$ be the set of the n peasants. For any $T \subset P$, we define $T^c := P - T$. Suppose that n is odd and let $x \in C(w)$. Let $T \subset P$ with $t = \frac{n+1}{2}$ and $\{i,j\} = \{1,2\}$.

Then $x(T \cup \{i\}) \geq w(T \cup \{i\}) = f(t) = f(\frac{n+1}{2})$ and similarly $x(T^c \cup \{j\}) \geq f(\frac{n-1}{2})$. Thus $x(N) = x(T \cup \{i\}) + x(T^c \cup \{j\}) \geq f(\frac{n+1}{2}) + f(\frac{n-1}{2}) = w(N)$ using (4.3). Because $x(N) = w(N)$, we can conclude that $x(T \cup \{i\}) = f(\frac{n+1}{2})$ and $x(T^c \cup \{j\}) = f(\frac{n-1}{2})$ for *any* $T \subset P$ with $t = \frac{n+1}{2}$ and $\{i,j\} = \{1,2\}$. From this, it follows that $x_1 = x_2$ and also $x_3 = \ldots = x_{n+2}$. Thus $C(w) = SC(w)$ if n is odd. The case n is even is treated similarly. We omit the proof of (ii) because it is similar to the proof of (iii).

(iii) Suppose that n is even. Let $x \in C(w)$. By (i), $x = (\alpha, \alpha, \beta, \ldots, \beta) \in \mathbb{R}^{n+2}$. Then $2\alpha + n\beta = w(N) = 2f(\frac{1}{2}n)$, $\alpha + \frac{1}{2}n + 1)\beta \geq f(\frac{1}{2}n + 1)$ and $\alpha + (\frac{1}{2}n - 1)\beta \geq f(\frac{1}{2}n - 1)$. It follows that $\delta^+ \leq \beta \leq \delta^-$. Thus $\beta \in co\{\delta^+, \delta^-\}$ and hence, $x \in co\{y,z\}$ where y and z are the vectors mentioned in statement (iii) of the theorem. It remains to show that $y, z \in C(w)$. But the vectors y and z satisfy the inequalities of (2.1) because of (1.5) and (1.3).

(iv) From the geometrical characterization of the nucleolus within the core (Maschler, Peleg, Shapley [5]) and the results in (ii) and (iii) it follows that the nucleolus of w has to be the midpoint of the line segment, which represents the core. □

Note that the formula of the nucleolus is described in terms of the marginal return(s) of the function f in the points nearest to the (rational) number $\frac{1}{2}n$. In case n is odd, the nucleolus payoff to any peasant equals the marginal return δ of f which is just his marginal contribution. In case n is even, this payoff equals half of the sum of the marginal returns δ^+ and δ^- of f, which is less than his marginal contribution δ^- (because $\delta^+ \leq \delta^-$). We remark that Chetty, Dasgupta, Raghavan [1] had already proved that δ^- (respectively δ^+) is an upper (lower) bound for the nucleolus payoff to any peasant in case n is even.

Now we determine the τ-value of the game w. First of all, we compare the gaps of the coalitions in case n is even.

Lemma 4.3. Let w be the game of (4.1) where f is concave and n is even. Then we have :

(i) $g^w(N) = n\delta^- - 2(f(n) - f(\frac{1}{2}n)) > 0$

(ii) $g^w(S) = s\delta^-$ whenever $1 \notin S$ and $2 \notin S$

(iii) $g^w(S) \geq g^w(N)$ whenever $\{1,2\} \subset S$.

(iv) Let $1 \in S$, $2 \notin S$ or $1 \notin S$, $2 \in S$. Then $g^w(S) \geq \frac{1}{2} g^w(N)$. Also $g^w(S) = \frac{1}{2} g^w(N)$ iff $s \in \{\frac{1}{2}n, \frac{1}{2}n + 1\}$.

The proof of this lemma is omitted because it consists of straightforward calculations. In order to calculate the τ-value, we look for the minimal gaps. In view of lemma 4.3 and definition (2.6), we have that $\lambda_1^w = \lambda_2^w = \frac{1}{2} g^w(N)$ and $\lambda_i^w = \min\{\delta^-, \frac{1}{2} g^w(N)\}$ for all $i \in N - \{1,2\}$. The next theorem follows then immediately from (2.8).

Theorem 4.4. Let w be the game of (4.1) where f is concave and n is even.

(i) If $\delta^- \geq \frac{1}{2} g^w(N)$, then $\lambda^w = \frac{1}{2} g^w(N)\, 1_{n+2}$ and $\tau(w) = b^w - (n+2)^{-1} g^w(N)\, 1_{n+2}$ where $1_{n+2} = (1, 1, \ldots, 1) \in \mathbb{R}^{n+2}$.

(ii) If $\delta^- \leq \frac{1}{2} g^w(N)$, then $\lambda^w = (\frac{1}{2} g^w(N), \frac{1}{2} g^w(N), \delta^-, \ldots, \delta^-)$ and

$\tau_i(w) = (g^w(N) + n\delta^-)^{-1} n\delta^- \delta^-$ for all $i \in N - \{1,2\}$,

$\tau_1(w) = \tau_2(w) = \frac{1}{2} w(N) - \frac{1}{2} (g^w(N) + n\delta^-)^{-1} n\delta^- n\delta^-$.

Let n be even. A peasant i can make a concession of at most his marginal contribution δ^- (since $\lambda_i^w \leq g^w(\{i\}) = b_i^w = \delta^-$), but his maximal concession λ_i^w will not exceed the maximal concession $\frac{1}{2} g^w(N)$ of a landowner. If δ^- does not exceed the amount $\frac{1}{2} g^w(N)$, then his maximal concession is δ^- and hence, his τ-value payoff is a part of δ^-, which is less than $(n+2)^{-1} n\delta^-$ (since $(g^w(N) + n\delta^-)^{-1} n\delta^- \leq (n+2)^{-1} n$ if $\delta^- \leq \frac{1}{2} g^w(N)$). If δ^- exceeds $\frac{1}{2} g^w(N)$, then the maximal concession of a peasant equals that of a landowner, which implies that the gap $g^w(N)$ between the utopia payoff $b^w(N)$ and the worth $w(N)$ is handed in equally by all players. The τ-value payoff to a peasant equals then $\delta^- - (n+2)^{-1} g^w(N)$ which is at least $(n+2)^{-1} n\delta^-$.

We conclude with some remarks. If n is odd, we can state results for the τ-value which are similar to those of theorem 4.4. In fact, we only need to replace δ^- by δ in theorem 4.4. Using the theorems 4.2 and 4.4, the following can be shown. For $n \in \{1,2\}$ we have that $\tau(w) = \eta(w) \in C(w)$. For $n > 2$ we have always that $\tau(w) \neq \eta(w)$ and hence $\tau(w) \notin C(w)$ in the odd case, but in the even case the τ-value *might* belong to the core. Finally, we remark that the formula of the Shapley value is omitted because it is somewhat complicated.

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